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# Remarks on Haar Systems*

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Let $D$ be any infinite compact subset of $\mathbf{R}$. An $n$-dimensional subspace $M$ of $C(D)$, $n \geqslant 1$, is a Haar space, if every nontrivial element of $M$ has at most $n - 1$ zeros in $D$. A system of functions $u_j \in C(D)$, $j = 1,..., n$, is a Haar system (Chebyshev system), if its span is an $n$-dimensional Haar space. A system $u_1 ,..., u_n$ is a Markov system, if $u_1 ,..., u_m$ is a Haar system for $m = 1,..., n$. (There is some lack of agreement in the literature as to which name should denote which system).

If $D$ is a proper subset of $\tilde{D} \subset \mathbf{R}$, and $\tilde{M}$ is a Haar space on $\tilde{D}$, then the restrictions of the elements of $\tilde{M}$ to $D$ form a Haar space $M \subset C(D)$. We call $\tilde{M}$ a proper extension of $M$.

We state the following two problems:

(1) Does every Haar space have a Markov basis (that is a basis which is a Markov system)?

(2) Does every Haar space have a proper extension?

Both questions have negative answers. In 1958, Volkov [3] gave an example of a Haar space on $[a, b]$, $a < b$, which cannot be extended to a Haar space on $[a, b] \cup \{c\}$, where $c$ is any point outside $[a, b]$, and hence has no proper extension at all. Later Kiefer and Wolfowitz [2] showed that the Volkov space has no Markov basis.

Both conjectures are true for $n = 1, 2$, and the Volkov example has dimension $n = 3$. This space consists of piecewise analytic functions, and the proof in [2] is not very appealing, although elementary. In the following we give a Haar system of analytic functions which provides negative answers for both questions. The proof is based on some simple geometric facts.

LEMMA 1. *Every Haar space contains a positive function.*

* This work was performed at the University of Minnesota in 1969/70.

*Proof.* See, e.g., [1, p. 28, Theorem 5.1].

LEMMA 2. *Let $M$ be an $n$-dimensional Haar space on $D$, and let $\tilde{M}$ be a proper extension on $\tilde{D} = D \cup D_1$, $D_1 \cap D = \varnothing$, where $D_1$ contains at least $n$ distinct points. Then $M$ has a basis, every permutation of which is a Markov system.*

*Proof.* Let $t_1, \ldots, t_n$ be $n$ distinct points of $D_1$, and let $u_1, \ldots, u_n$ be a basis of $\tilde{M}$. The matrix $(u_j(t_k))$ is nonsingular since $\tilde{M}$ is a Haar space. Let $(p_{jk})$ be its inverse. The system $v_1, \ldots, v_n$, where $v_j = \sum p_{jk} u_k$, satisfies $v_j(t_l) = \delta_{jl}$, $j, l = 1, \ldots, n$.

Now any nontrivial linear combination of $m$ different $v_j$ has at least $n - m$ zeros in $D_1$ and at most $n - 1$ zeros in $\tilde{D}$, hence at most $m - 1$ zeros in $D$.

Similarly one can prove the following.

LEMMA 3. *Let $M$ be an $n$-dimensional Haar space on $D$, and let $\tilde{M}$ be a proper extension on $\tilde{D} = D \cup D_1$, $D_1 \cap D = \varnothing$, where $D_1$ contains $q$ distinct points, $1 \leqslant q \leqslant n - 1$. Then there are Haar spaces $M_p$, $\dim M_p = p$, $p = n - q, \ldots, n - 1$, with $M \supset M_{n-1} \supset \cdots \supset M_{n-q}$.*

According to Lemma 1 the existence of $M_2$ implies the existence of $M_1$, hence $n - 2$ points in $D_1$ ensure the existence of $M \supset M_{n-1} \supset \cdots \supset M_2 \supset M_1$.

Let $M$ be a subspace of $C[a, b]$ with basis $u_1, \ldots, u_n$. We introduce an $n$-dimensional euclidean space $\mathbf{R}^n$ and interpret $M$ as a curve $g$ in $\mathbf{R}^n$ by $g(t) = (u_1(t), \ldots, u_n(t))$.

Obviously we have (with appropriate counting of multiple points)

LEMMA 4. *$M$ is a Haar space, if every hyperplane containing the origin intersects $g$ in at most $n - 1$ points.*

Since a Haar space contains a positive function, from now on we can assume $u_1(t) \equiv 1$. Then the image of $g$ is in the hyperplane $u_1 = 1$.

LEMMA 5. *Let $M$ be a Haar space. The following statements are equivalent:*

(1) *$M$ has a proper extension.*

(2) *$M$ contains a Haar space of dimension $n - 1$.*

(3) *There is a point $P \in \mathbf{R}^n$, such that every hyperplane containing $P$ and the origin intersects $g$ in at most $n - 2$ points.*

*Proof.* The equivalence of (1) and (3) is quite obvious in view of Lemma 4, and from Lemma 3 we infer that (1) implies (2). Now assume (2) with a Haar space $H \subset M$, $\dim H = n - 1$. The elements $\alpha_1 u_1 + \cdots + \alpha_n u_n$ of $H$ correspond to the hyperplanes $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$ in $\mathbf{R}^n$, and the coefficients

$\alpha_j$ are subjected to one nontrivial linear condition $\alpha_1 p_1 + \cdots + \alpha_n p_n = 0$; that is, all hyperplanes corresponding to elements of $H$ contain the point $P = (p_1, \ldots, p_n)$, and every hyperplane containing $P$ represents an element of $H$. Since $H$ is a Haar space, each hyperplane intersects $g$ in at most $n - 2$ points.

For $n = 3$, using $u_1 = 1$, (3) can be restated as

(4) In the plane $u_1 = 1$ there is a point $P'$ (possibly at infinity) such that every member of the line bundle containing $P'$ intersects $g$ in at most one point.

THEOREM 6. *The system of functions on* $[0, 3\pi/2]$ *defined by*

$$u_1(t) = 1,$$
$$u_2(t) = \cos t + \sin t - \tfrac{1}{2} \cos 2t,$$
$$u_3(t) = \cos t - \sin t + \tfrac{1}{2} \sin 2t$$

*generates a Haar space without proper extension and without a 2-dimensional Haar subspace.*

*Proof.* Another basis of the space is given by

$$v_1 = u_1, \qquad v_2 = 2u_3 - u_1, \qquad v_3 = 2u_2.$$

We have

$$\begin{aligned} v_2(t) &= 2\cos t - 2\sin t + 2\sin t\cos t - \cos^2 t - \sin^2 t \\ &= (2 - \cos t + \sin t)(\cos t - \sin t) \\ &= 2(\sqrt{2} - \tfrac{1}{2}\sqrt{2}\cos t + \tfrac{1}{2}\sqrt{2}\sin t)(\tfrac{1}{2}\sqrt{2}\cos t - \tfrac{1}{2}\sqrt{2}\sin t) \\ &= 2\left(\sqrt{2} - \cos\left(t + \frac{\pi}{4}\right)\right)\cos\left(t + \frac{\pi}{4}\right), \end{aligned}$$

and similarly

$$v_3(t) = 2\left(\sqrt{2} - \cos\left(t + \frac{\pi}{4}\right)\right)\sin\left(t + \frac{\pi}{4}\right).$$

Substituting $s = t + \pi/4$, we get the equivalent system

$$v_1(s) = 1,$$
$$v_2(s) = 2(\sqrt{2} - \cos s)\cos s, \ (\pi/4) \leqslant s \leqslant (7\pi/4),$$
$$v_3(s) = 2(\sqrt{2} - \cos s)\sin s.$$

The image of $g$ is now an arc of a Pascal limaçon $r(s) = 2\sqrt{2} - 2\cos s$, located in the plane $u_1 = 1$. The values $s = \pi/4$, $s = 7\pi/4$ of the parameter correspond to the two points of the limaçon having a common tangent (the image of $g$ is the intersection of the limaçon and the boundary of its convex hull). Every straight line in $u_1 = 1$ intersects $g$ in at most two points, hence $v_1, v_2, v_3$ span a Haar space. But property (4) does not hold. Hence this space has no proper extension, does not contain a Haar space of dimension 2 and has no Markov basis.

There are still some open questions, e.g., whether the existence of a Markov basis implies the existence of a proper extension with $n - 2$ additional points (a counterexample would require $n \geqslant 4$), and what are the conditions for the existence of a proper extension with $D$ being an interval.

## References

1. S. J. Karlin and W. J. Studden, "Chebyshev Systems: With Applications in Analysis and Statistics," Wiley, New York, 1966.
2. J. Kiefer and J. Wolfowitz, On a theorem of Hoel and Levine on extrapolation designs, *Ann. Math. Statist.* **36** (1965), 1627–1655.
3. V. I. Volkov, Some properties of Chebyshev systems, *Kalinin. Gos. Ped. Inst. Učen. Zap.* **26** (1958), 41–48.
4. A. B. Nemeth, Transformations of the Chebyshev systems, *Mathematica (Cluj)* **8** (1966), 315–333.
5. A. B. Nemeth, About the extension of the domain of definition of the Chebyshev systems defined on intervals of the real axis, *Mathematica (Cluj)* **11** (1969), 307–310.
6. R. Zielke, On transforming a Tchebyshev-system into a Markov-system, *J. Approximation Theory*, to appear.
7. R. Zielke, A remark on periodic Tchebyshev systems, *Manuscripta mathematica* **7** (1972), 325–329.